

# Chapter 2

## The Maxwell Equations

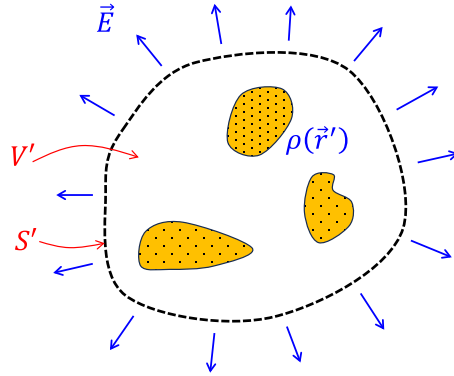
### 2.1 Maxwell Equations in Vacuum

Maxwell's equations, the fundamental equations governing electromagnetism, describe how electric and magnetic fields are generated and interact with charges and currents. They are formulated in both differential and integral forms and are foundational in fields such as electrodynamics, optics, and electrical engineering. The differential form of Maxwell's equations is given below,

$$\left\{ \begin{array}{ll} \nabla \cdot \vec{D} = \rho & \text{Gauss law for E – field} \\ \nabla \cdot \vec{B} = 0 & \text{Gauss law for B – field} \\ \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} & \text{Ampere's law} \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 & \text{Faraday's law} \end{array} \right. , \quad (2.1)$$

where  $\vec{D}$  is the electric displacement field, which accounts for the effects of both free and bound charges in materials;  $\vec{B}$  is the magnetic field, representing the density of magnetic flux;  $\vec{E}$  is the electric field, the force per unit charge experienced by a static electric charge; and  $\vec{H}$  is the magnetic field intensity, which relates to  $\vec{B}$  through the material's magnetic properties;  $\rho$  is the free charge density, representing the amount of charge per unit volume; and  $\vec{J}$  is the current density, representing the current per unit area. The corresponding integration format are

$$\left\{ \begin{array}{l} \oiint \vec{D} \cdot \hat{n} dS' = q \\ \oiint \vec{B} \cdot \hat{n} dS' = 0 \\ \oint_L \vec{H} \cdot d\vec{l} = \iint_S \left[ \vec{J} + \frac{\partial \vec{D}(t)}{\partial t} \right] \cdot \hat{n} dS' \\ \oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \hat{n} dS' \end{array} \right. \quad (2.2)$$



**Fig. 2.1** The charge distributed in an enclosed surface and the electric flux.

### 2.1.1 Gauss law for electric field $\vec{E}$

The Gauss law for the electric field  $\vec{E}$  is expressed as the total electric flux  $\Phi_E$  coming out from a closed surface as shown in **Figure 2.1** is determined by the total charge  $Q$  enclosed in the closed surface, i.e.,

$$\oiint \vec{E} \cdot \hat{n} dS' = \frac{q}{\epsilon_0}. \quad (2.2)$$

Here the total charge  $q$  can be written as,

$$q = \iiint_{V'} \rho(\vec{r}') dV', \quad (2.3)$$

where  $\rho(\vec{r}')$  is the bulk charge density as indicated in **Figure 2.1**. Such a charge density can be a continuous charge distribution inside the enclosed surface or a discretized charge distribution under consideration. For either case,  $\rho(\vec{r}')$  can be expressed as a continuum function of  $\vec{r}'$  (for a discretized distribution, the function  $\rho(\vec{r}')$  may contain delta-function (s)). According to Gauss theorem of multi-variable calculus,

$$\oiint \vec{E} \cdot \hat{n} dS' = \iiint_V \nabla \cdot \vec{E} dV'. \quad (2.4)$$

By comparing **Equation 2.4** with **Equation 2.3**, and noting that both equations hold for any arbitrary enclosed surface, we arrive at the relationship

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \quad (2.5)$$

In vacuum, Maxwell introduces the displacement field  $\vec{D} = \epsilon_0 \vec{E}$ , leading to,

$$\nabla \cdot \vec{D} = \rho. \quad (2.6)$$

This corresponds to the first equation in **Equation 2.1**.

to describe charge distributions under different geometric configurations, we define specific charge densities. For a bulk charge distribution, the charge density  $\rho(\vec{r}')$  represents the charge per unit volume. The total charge  $q$  within a bounded surface can then be calculated using **Equation 2.3**. If the charges are confined to a surface, a surface charge density  $\sigma(\vec{r}'_s)$  (in Coulombs per unit area) is used. The total charge is determined as,

$$q = \iint_S \sigma(\vec{r}'_s) dS'. \quad (2.7)$$

For charges distributed along a line, the linear charge density  $\lambda(\vec{r}'_l)$  (in Coulombs per unit length) is applicable, and the total charge is given by

$$q = \int_L \lambda(\vec{r}'_l) dl. \quad (2.8)$$

For multiple discrete point charges  $q_j$  located in space, the charge density can be expressed using the Dirac delta function as,

$$\rho(\vec{r}') = \sum_{j=1}^N q_j \delta(\vec{r} - \vec{r}_j). \quad (2.9)$$

### 2.1.2 Gauss law for magnetic field $\vec{B}$

A moving charge  $q$  with a velocity  $\vec{v}$  in a uniform magnetic field  $\vec{B}$  experiences a magnetic force  $\vec{F}_B$ ,

$$\vec{F}_B = q\vec{v} \times \vec{B}. \quad (2.10)$$

Alternatively, a magnetic field  $\vec{B}$  is generated by moving charges or currents, where the current  $I$  is defined as,

$$I = \frac{dQ}{dt} = \iint_S \vec{j} \cdot \hat{n} dS', \quad (2.11)$$

where  $I$  is the rate of the change of the electric charge across a surface. Microscopically, the current is determined by the current density  $\vec{j}$ . In magnetism, the following principles are important:

1) Lorentzian force: the total force  $\vec{F}_L$  acting on a charged particle in the presence of both E- and B-fields is called the Lorentzian force

$$\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B}). \quad (2.12)$$

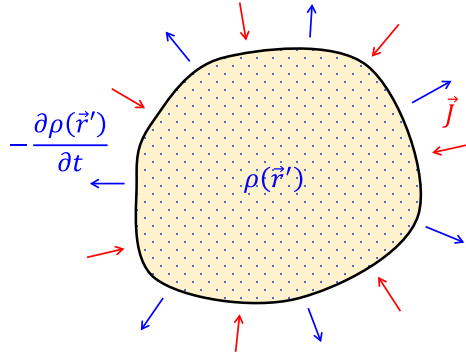
2) Conservation of charges: The net current passing through an enclosed surface equals the total rate of change of charge within the enclosed volume, as

charges cannot be created or destroyed in vacuum. As shown in **Figure 2.2**, the total current  $I$  flowing into a closed surface (red arrows in **Figure 2.2**) is,

$$I_{in} = \oint_S \vec{J} \cdot \hat{n} dS' = \iiint_V \nabla \cdot \vec{J} dV'. \quad (2.13)$$

The outward current  $I_{out}$  (blue arrows in **Figure 2.2**), due to the change in total charge within the volume, is,

$$I_{out} = -\frac{dq}{dt} = -\frac{d}{dt} \iiint_V \rho(\vec{r}') dV' = -\iiint_V \frac{\partial \rho(\vec{r}')}{\partial t} dV'. \quad (2.14)$$



**Fig. 2.2** The current density and the change of charge density around a closed surface.

By conservation of charge,  $I_{in} = I_{out}$ , leading to  $\iiint_V \nabla \cdot \vec{J} dV' = -\iiint_V \frac{\partial \rho(\vec{r}')}{\partial t} dV'$ . Since this expression is valid for any situation, it simplifies to

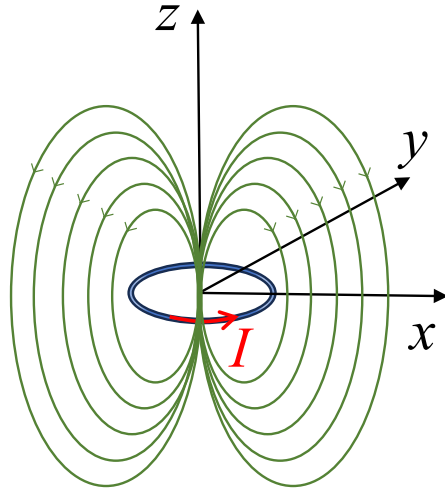
$$\nabla \cdot \vec{J} + \frac{\partial \rho(\vec{r}')}{\partial t} = 0. \quad (2.25)$$

The current density depends on the nature of the charge distribution. For point charges,

$$\vec{J}(\vec{r}') = \sum_{j=1}^N q_j \vec{v}_j \delta(\vec{r}' - \vec{r}_j). \quad (2.26)$$

For continuous charge distributions, assuming a constant velocity  $\vec{v}$  of electric charges flowing through a cross-section/cross-line/cross-point for a bulk/surface/line current distribution, then the respective current density  $\vec{J}(\vec{r}')$  can be expressed as,

$$\begin{cases} \vec{J}(\vec{r}') = \rho(\vec{r}')\vec{v}, & \text{for a bulk} \\ \vec{J}(\vec{r}') = \sigma(\vec{r}')\vec{v}, & \text{for a surface} \\ \vec{J}(\vec{r}') = \lambda(\vec{r}')\vec{v}, & \text{for a line} \end{cases} \quad (2.27)$$



**Fig. 2.3** The magnetic field lines generated by a magnetic dipole.

3) *The source of magnetic field:* The magnetic field  $\vec{B}$  arises from magnetic dipoles (see **Figure 2.3**), currents, moving charges, or spins. Since no magnetic monopoles have been observed experimentally,

$$\oiint \vec{B} \cdot \hat{n} dS' = 0, \tag{2.28}$$

which, by Gauss's theorem, implies,

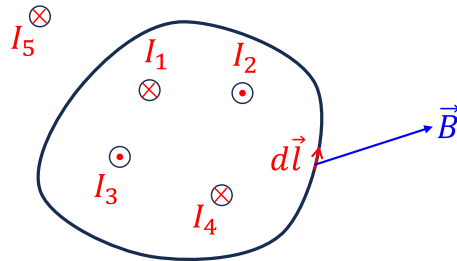
$$\nabla \cdot \vec{B} = 0. \tag{2.29}$$

4) *The Biot-Savart law:* The magnetic field produced by a current is represented by the Biot-Savart law (more details see **Chapter 9**),

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \tag{2.30}$$

For a current density  $\vec{J}(\vec{r}')$ , **Equation 2.30** can be written as

$$\vec{B} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'. \tag{2.31}$$



**Fig. 2.4** Loop integration of the magnetic field.

From the Biot-Savart law, one can derive Ampere's law for magnetostatics.

### 2.1.3 Ampere's law

The line integral of the magnetic field  $\vec{B}$  around a closed loop is given by

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \sum_{in} I_i. \quad (2.32)$$

In **Figure 2.4**, the summation  $\sum_{in} I_i$  includes currents  $I_1, I_2, I_3, I_4$  within the integration loop. Currents outside the loop, such as  $I_5$ , do not contribute. For a continuous charge distribution, this can be expressed as

$$\sum_{in} I_i = \iint_S \vec{j} \cdot \hat{n} dS'. \quad (2.33)$$

Using Stokes' theorem, we can rewrite the line integral as,

$$\oint_L \vec{B} \cdot d\vec{l} = \iint_S \nabla \times \vec{B} \cdot \hat{n} dS'. \quad (2.34)$$

which leads to,

$$\nabla \times \vec{B} = \mu_0 \vec{j}. \quad (2.35)$$

#### Issue with Magnetostatic Ampère's Law

A limitation of **Equation 2.32** arises when considering the charging of a capacitor. As illustrated in **Figure 2.5**, when a parallel-plate capacitor is connected to a battery, a time-dependent current  $I(t)$  flows as the capacitor charges. Let us assume that each capacitor plate is extremely thin. To examine this scenario, consider two specific loops:

1. Loop  $L_L$ : This loop surrounds the charging wire near the left plate. For this loop

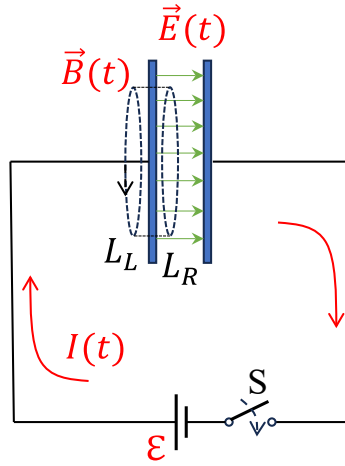
$$\oint_{L_L} \vec{B} \cdot d\vec{l} = \mu_0 I(t). \quad (2.36)$$

Here, the current  $I(t)$  generates a magnetic field around the wire.

2. Loop  $L_R$ : This loop is located between the two capacitor plates, very close to the left plate. For this loop, Ampère's law (**Equation 2.32**) gives,

$$\oint_{L_R} \vec{B} \cdot d\vec{l} = 0. \quad (2.37)$$

According to **Equation 2.32**, there is no magnetic field between the plates, which contradicts the continuity of the magnetic field near the left plate.



**Fig. 2.5** The magnetic field near a charging capacitor.

To resolve this, consider a small cylindrical Gaussian surface crossing the left plate. On the left side of the plate, the electric field is zero, while on the right side, it is constant. From Gauss's law,

$$E(t)A = \frac{\sigma(t)A}{\epsilon_0}, \quad (2.38)$$

where  $A$  is the plate area and  $\sigma(t)$  is the surface charge density. This gives

$$\sigma(t) = \epsilon_0 E(t). \quad (2.39)$$

The changing surface charge density leads to an effective current density,

$$J(t) = \frac{d\sigma(t)}{dt} = \epsilon_0 \frac{dE(t)}{dt}. \quad (2.40)$$

This current density, caused by the time-varying electric field  $\vec{E}(t)$ , is called the displacement current. Ampère's law is thus modified to include this displacement current,

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 \sum_{in} I_i + \mu_0 \epsilon_0 \iint_S \frac{\partial \vec{E}(t)}{\partial t} \cdot \hat{n} dS'. \quad (2.41)$$

This can also be written as,

$$\oint_L \vec{H} \cdot d\vec{l} = \iint_S \left[ \vec{J} + \frac{\partial \vec{D}(t)}{\partial t} \right] \cdot \hat{n} dS', \quad (2.42)$$

where  $\vec{H} = \frac{\vec{B}}{\mu_0}$ , and the differential form becomes

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}(t)}{\partial t}. \quad (2.43)$$

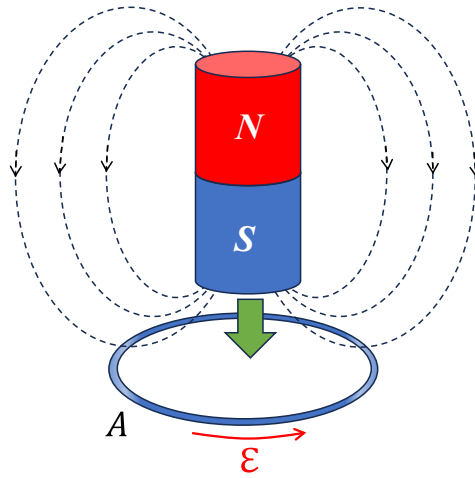


Fig. 2.6 A moving bar magnet towards a conducting loop.

#### 2.1.4 Faraday's law

A changing magnetic flux through a conductor loop induces an electromotive force (EMF)  $\mathcal{E}$ , expressed as

$$\mathcal{E} = -N \frac{d\Phi_B}{dt}. \quad (2.44)$$

where  $N$  is the total number of loops, and the magnetic flux is defined as

$$\Phi_B = \iint_S \vec{B} \cdot \hat{n} dS'.$$

The EMF can also be expressed as

$$\mathcal{E} = \oint_L \vec{E} \cdot d\vec{l}.$$

Using Stokes' theorem, this becomes

$$\oint_L \vec{E} \cdot d\vec{l} = \iint_S \nabla \times \vec{E} \cdot \hat{n} dS'. \quad (2.45)$$

For a single loop ( $N = 1$ ), the relationship simplifies to

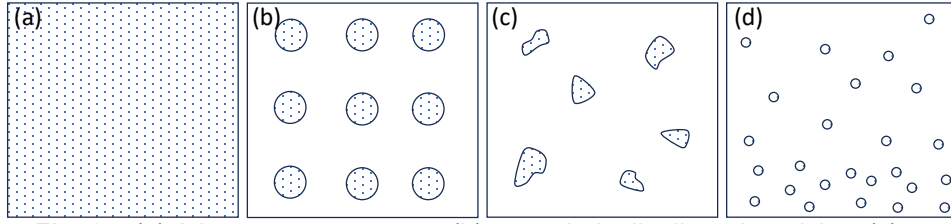
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (2.46)$$

This is Faraday's law, which describes how a time-varying magnetic field generates an electric field.

## 2.2 Maxwell Equations in Matter

In most practical scenarios, electric and magnetic fields exist within matter, requiring consideration of Maxwell's equations in material media. From a microscopic perspective, the E- and B-fields in matter fundamentally behave as





**Fig. 2.7** (a) A homogenous matter; (b) a regularly distributed particles; (c) a randomly distributed irregular particles; (d) a nonuniformly distributed particles.

they do in a vacuum. However, their relative amplitudes, directions, and phases are altered due to interactions with the material. While the original Maxwell's equations remain valid, the relationships between  $\vec{E}$ ,  $\vec{D}$ ,  $\vec{B}$ , and  $\vec{H}$  are modified,

$$\begin{cases} \vec{D} = \vec{D}(\vec{E}, \vec{B}) \\ \vec{H} = \vec{H}(\vec{E}, \vec{B}) \end{cases} \quad (2.47)$$

These are known as constitutive relationships. For most linear, isotropic materials, the relationships can be expressed as,

$$\begin{cases} \vec{D} = \epsilon_0 \vec{E} + \vec{P} \\ \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \end{cases} \quad (2.48)$$

and the current density is given by

$$\vec{J} = \vec{J}(\vec{E}, \vec{B}) = \sigma_c \vec{E}. \quad (2.49)$$

where  $\vec{P}$  and  $\vec{M}$  are called the polarization and magnetization of the material, and  $\sigma_c$  is the material's conductivity. **Equation 2.49** is commonly referred to as the microscopic Ohm's law.

Materials can be categorized based on their behavior in response to electric and magnetic fields.

### 2.2.1 Linear and nonlinear matters

For linear materials, the relationship between  $\vec{D}$  and  $\vec{E}$ , as well as between  $\vec{B}$  and  $\vec{H}$ , are linear

$$\begin{cases} \vec{D} = \epsilon \epsilon_0 \vec{E} \\ \vec{B} = \mu \mu_0 \vec{H} \end{cases} \quad (2.50)$$

where  $\epsilon$  and  $\mu$  are the relative dielectric and magnetic permeabilities of the material.

For nonlinear materials, the relationships are more complex and include higher-order terms,

$$\begin{cases} \vec{D} = \epsilon \epsilon_0 (\vec{E} + \chi_{E2} \vec{E} \vec{E} + \chi_{E3} \vec{E} \vec{E} \vec{E} + \dots) \\ \vec{H} = \mu \mu_0 (\vec{B} + \chi_{M2} \vec{B} \vec{B} + \chi_{M3} \vec{B} \vec{B} \vec{B} + \dots) \end{cases} \quad (2.51)$$

where  $\chi_{E2}$  and  $\chi_{E3}$  are the 2<sup>nd</sup> and 3<sup>rd</sup> order electric nonlinearity tensors, while  $\chi_{H2}$  and  $\chi_{H3}$  are the 2<sup>nd</sup> and 3<sup>rd</sup> order magnetic nonlinearity tensors. These nonlinearities describe how the material responds to stronger fields or exhibits phenomena like harmonic generation.

### 2.2.2 Homogenous and inhomogeneous matters

In a homogeneous material (**Figure 2.7A**), atoms or molecules are uniformly distributed throughout the entire volume or surface. For linear and homogeneous materials, the permittivity  $\varepsilon$  and permeability  $\mu$  are constant and do not vary with location.

If the material consists of regularly distributed particles or units (**Figure 2.7B**), whose size and spacing are comparable to the wavelength of electromagnetic waves, strong electromagnetic coupling between the units can significantly alter  $\varepsilon$  and  $\mu$ . These materials are called metamaterials. In such cases, the arrangement, size, and shape of the particles play a crucial role in determining the effective  $\varepsilon$  and  $\mu$ . Furthermore, the parameters inside, near, and between the particles can differ significantly

For materials where the particles or units are distributed randomly (**Figure 2.7C**), with random size and shape distributions, the material is termed a random medium. Statistically, such media can be treated as homogeneous, with effective  $\varepsilon$  and  $\mu$  values determined by the spatial and morphological distribution of the particles. However, within the random medium, the local values of  $\varepsilon$  and  $\mu$  vary significantly near and inside the particles.

When particles or units are non-uniformly distributed (**Figure 2.7D**), both  $\varepsilon$  and  $\mu$  become functions of position  $\vec{r}'$ , leading to spatially varying fields

$$\vec{D}(\vec{r}') = \varepsilon(\vec{r}')\varepsilon_0\vec{E}(\vec{r}'), \quad \vec{B}(\vec{r}') = \mu(\vec{r}')\mu_0\vec{H}(\vec{r}'). \quad (2.52)$$

In such cases, the spatial variation must be considered. However, if the variations are uniform on a macroscopic scale, effective medium theories, such as mean-field approximations, can be applied.

### 2.2.3 Isotropic and anisotropic matters

Isotropic materials exhibit uniform properties in all directions, meaning their response to electric or magnetic fields does not depend on the direction of the applied field. For isotropic materials,  $\varepsilon$  and  $\mu$  are scalars, simplifying Maxwell's equations to **Equation 2.50**.

Anisotropic materials have direction-dependent properties, often resulting from their lattice structure (e.g., 2D materials, nanotubes) or morphology (e.g., nanorod arrays, patterned surfaces). Their response to external fields is described using tensors,

$$\vec{D}(\vec{r}') = \vec{\varepsilon}(\vec{r}')\varepsilon_0\vec{E}(\vec{r}'), \quad \vec{B}(\vec{r}') = \vec{\mu}(\vec{r}')\mu_0\vec{H}(\vec{r}'). \quad (2.53)$$

Here  $\vec{\epsilon}$  and  $\vec{\mu}$  are the second order tensors,

$$\vec{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}. \quad (2.54)$$

For homogeneous, linear, and isotropic materials, the tensors simplify to,

$$\vec{\epsilon} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}. \quad (2.55)$$

In anisotropic materials,  $\vec{D}$  and  $\vec{B}$  may not align with  $\vec{E}$  and  $\vec{H}$ , respectively, reflecting directionally dependent responses.

### 2.2.4 Stimulate or active matters

Stimulated or active materials respond dynamically to external stimuli such as electric, magnetic, mechanical, or optical fields. Their electromagnetic properties can be tuned, enabling complex interactions with electromagnetic waves. Three notable effects are discussed below.

#### 1. Electro-optical (EO) effect

The EO effect alters a material's optical properties under an applied electric field. Examples include:

- Pockels effect (linear): The refractive index change  $\Delta\epsilon_{ij}$  is proportional to the field,

$$\Delta\epsilon_{ij} = \sum_{k=1}^3 r_{ijk} E_k. \quad (2.56)$$

where  $r_{ijk}$  is the EO coefficient tensor specific to the material, and  $E_k$  is the component of applied electric field. The Pockels effect is absent in materials with centrosymmetry, meaning it is primarily observed in crystals lacking a center of symmetry, such as lithium niobate (LiNbO<sub>3</sub>), potassium titanyl phosphate (KTP), and gallium arsenide (GaAs). The Pockels effect is commonly used in EO modulators, which control the phase or amplitude of light passing through the material. These modulators are essential in telecommunications for modulating laser signals in optical fiber systems.

- Kerr effect (quadratic): The change in the refractive index  $\Delta\epsilon_{ij}$  is proportional to the square of the applied electric field,

$$\Delta\epsilon_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 K_{ijkl} E_k E_l, \quad (2.57)$$

where  $K_{ijkl}$  is the Kerr coefficient tensor, and  $E_k$  and  $E_l$  are components of the electric field. Unlike the Pockels effect, the Kerr effect can occur in centrosymmetric materials, and it is observed in materials such as nitrobenzene

and CS<sub>2</sub> (carbon disulfide). It is utilized in Kerr cells, which are used in high-speed optical shutters, modulators, and Q-switching devices in lasers. It is also the basis for optical Kerr microscopy and ultrafast pulse generation.

The electrochromic effect is a phenomenon where the optical properties, particularly the absorption spectrum of a material, change in response to an applied electric field or current, often leading to a visible change in color. This effect is typically reversible, and the change in dielectric properties can be described as,

$$\Delta\varepsilon(\lambda, E) = \varepsilon(\lambda, E) - \varepsilon(\lambda, 0). \quad (2.58)$$

where  $\varepsilon(\lambda, E)$  is the dielectric function dependent on the wavelength  $\lambda$  and the applied electric field  $E$ . Typical electrochromic materials include transition metal oxides such as tungsten oxide (WO<sub>3</sub>) and organic compounds like viologens. The electrochromic effect is widely used in smart windows, which can change transparency to control the amount of light and heat passing through, thereby improving energy efficiency in buildings. It is also used in rearview mirrors for automobiles, displays, and low-power electronic paper.

## 2. The magneto-optical (MO) effect

The MO effect refers to the interaction between light and a material in the presence of a magnetic field, leading to changes in the material's optical properties, such as its refractive index, absorption, and polarization state. This effect arises due to the coupling between the magnetic field and the electronic structure of the material, which can alter how the material interacts with electromagnetic waves. The magneto-optical effect is often characterized by changes in the dielectric tensor, which becomes anisotropic and complex, depending on the direction and magnitude of the applied magnetic field.

The Faraday effect is a MO phenomenon where the polarization plane of linearly polarized light rotates as it passes through a material under the influence of a magnetic field parallel to the direction of light propagation. The rotation angle  $\theta_F$  is proportional to the magnetic field strength  $B$ , the length of the medium  $L$ , and the Verdet constant  $V$  of the material,

$$\theta_F = VBL. \quad (2.59)$$

This effect can be understood through the magneto-optical modification of the dielectric tensor. In an isotropic material, the dielectric tensor in the presence of a magnetic field (in  $z$ -direction) becomes,

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon & i\varepsilon_{MO} & 0 \\ -i\varepsilon_{MO} & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \quad (2.60)$$

with  $\varepsilon_{MO} \propto \vec{B}$ , where  $\vec{B}$  is the external applied magnetic field, which introduces off-diagonal components causing the rotation of polarization. The Faraday effect is observed in materials like terbium gallium garnet (TGG), yttrium iron garnet (YIG), and flint glass. It is widely used in optical isolators and circulators, which

are essential components in laser systems and optical communication networks to prevent back-reflected light from interfering with the source.

The magneto-optical Kerr effect (MOKE) occurs when the polarization of light reflected from a magnetized surface is altered. The effect can manifest in three main forms: polar, longitudinal, and transverse, depending on the orientation of the magnetic field relative to the light's incidence.

- Polar MOKE: Observed when the magnetic field is perpendicular to the surface.
- Longitudinal MOKE: Occurs when the magnetic field is parallel to both the surface and the plane of incidence.
- Transverse MOKE: Seen when the magnetic field is parallel to the surface but perpendicular to the plane of incidence.

The change in polarization due to MOKE can be described by a complex Kerr rotation angle  $\theta_K + i\eta_K$ , where  $\theta_K$  is the rotation of the polarization plane and  $\eta_K$  is the ellipticity induced,

$$\theta_K + i\eta_K = \frac{\varepsilon_{ij}}{\varepsilon_{ii}}, \quad (2.61)$$

where  $\varepsilon_{ij}$  is the off-diagonal component of the dielectric tensor induced by the magnetic field, and  $\varepsilon_{ii}$  is the diagonal component. MOKE is prominent in magnetic materials such as iron (Fe), cobalt (Co), and nickel (Ni), as well as in magnetic thin films and multilayers. It is used in magnetic storage technology, such as in magneto-optical drives, and in the characterization of magnetic thin films and nanostructures through MOKE microscopy.

### 3. Chiral material

Chiral materials are a class of substances that lack mirror symmetry, meaning they cannot be superimposed on their mirror image. This inherent asymmetry, known as chirality, leads to unique optical properties, particularly in how they interact with polarized light. Chiral materials can rotate the polarization plane of light (optical activity) or exhibit different absorption of left- and right-circularly polarized light (circular dichroism). These properties make chiral materials essential in various scientific and industrial applications, including optics, pharmaceuticals, and materials science. The constitutive relationship for chiral materials is more complex than for non-chiral materials because it includes terms that account for the coupling between the electric and magnetic fields,

$$\begin{cases} \vec{D} = \varepsilon\varepsilon_0\vec{E} + i\xi\vec{H} \\ \vec{B} = \frac{1}{\mu\mu_0}\vec{H} - i\xi\vec{E}' \end{cases} \quad (2.62)$$

where  $\xi$  is the chirality parameter. The term  $i\xi$  introduces a coupling between the electric and magnetic fields, which is responsible for the chiral optical effects, such

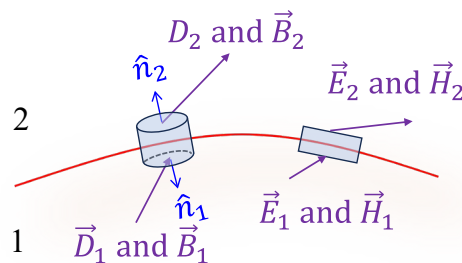
as optical activity and circular dichroism. When light propagates through a chiral material, its behavior differs significantly from propagation in non-chiral media. Two key effects are observed: First, optical rotation (optical activity). As linearly polarized light passes through a chiral material, the polarization plane rotates. This rotation is proportional to the thickness of the material and the strength of its optical activity. The rotation angle  $\theta$  can be described by,

$$\theta = \frac{\omega\xi L}{c}, \quad (2.63)$$

where  $\omega$  is the angular frequency of light and  $L$  is the length of the material. Second, circular dichroism. Chiral materials absorb left- and right-circularly polarized light differently, leading to an elliptically polarized transmitted beam. This differential absorption results from the different interactions between the material and the two circular polarizations, which is quantified by the difference in their extinction coefficients.

Many biomolecules, such as proteins and sugars, are inherently chiral and exhibit optical activity. For example, glucose and other saccharides are well-known for their ability to rotate the plane of polarized light. Chiral liquid crystals, such as those used in liquid crystal displays (LCDs), are engineered to manipulate light through their twisted molecular structure. Engineered chiral metamaterials are designed to exhibit strong chiral properties, including negative refraction and enhanced circular dichroism, by carefully arranging sub-wavelength structures.

The ability of chiral materials to rotate the polarization of light is used in polarimeters to measure the concentration of chiral substances in solutions, such as sugars and pharmaceuticals. In addition, chiral materials are used in devices like circular polarizers and optical isolators, which rely on manipulating light's polarization state. Furthermore, in the pharmaceutical industry, the chirality of drugs is critical because different enantiomers (mirror-image forms) of a molecule can have drastically different biological effects. Finally, the engineered chiral metamaterials can be used in applications such as advanced sensing, imaging, and controlling light propagation in novel ways, including cloaking and negative refraction.



**Fig. 2.8** (a) A homogenous matter; (b) a regularly distributed particles; (c) a randomly distributed irregular particles

### 2.3 Boundary Conditions

Many E- and B-field related problems occur at two materials interfaces. Certain components of E- and B-fields at the boundary shall be continuous at the material interface. As shown in **Figure 2.8**, using the integration forms of maxwell equations, construct a small cylindrical Gaussian surface and rectangular loop across boundary, one has,

$$\begin{cases} \oiint_S \vec{D} \cdot \hat{n} dS' = \iiint_V \rho(\vec{r}') dV' \\ \oiint_S \vec{B} \cdot \hat{n} dS' = 0 \end{cases} \quad (2.64)$$

We have,

$$\begin{cases} (\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_2 = \sigma \\ (\vec{B}_2 - \vec{B}_1) \cdot \hat{n}_2 = 0 \end{cases} \quad (2.65)$$

i.e., normal components of  $\vec{D}$  field may not be continuous if there are free surface charge distribution  $\sigma$ ; while for the magnetic field  $\vec{B}$ , its normal components are continuous across the interface. For the loop, using the loop integration equations,

$$\begin{cases} \oint_L \vec{H} \cdot d\vec{l} = \iint_S \left[ \vec{J} + \frac{\partial \vec{D}(t)}{\partial t} \right] \cdot \hat{n} dS' \\ \oint_L \vec{E} \cdot d\vec{l} = - \iint_S \frac{\partial \vec{B}(t)}{\partial t} \cdot \hat{n} dS' \end{cases} \quad (2.66)$$

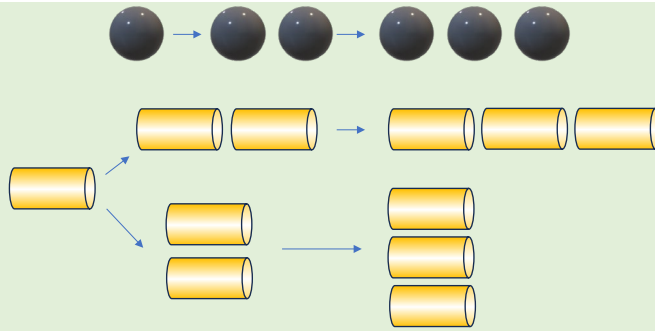
the tangent components of the fields can be written as,

$$\begin{cases} (\vec{E}_2 - \vec{E}_1) \times \hat{n}_2 = 0 \\ ((\vec{H}_2 - \vec{H}_1) \times \hat{n}_2 = \vec{K}(\vec{r}_s)) \end{cases} \quad (2.67)$$

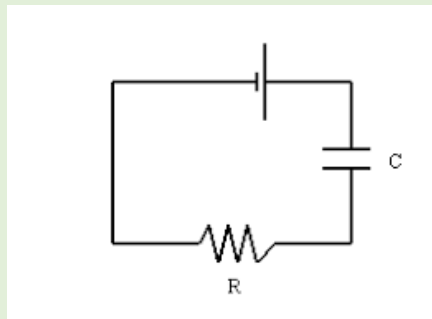
where  $\vec{K}(\vec{r}_s)$  is the surface current density at the boundary. **Equation 2.67** shows that at the boundary, the tangent components of the electric field  $\vec{E}$  are continuous, and the magnetic intensity  $\vec{H}$  may not be continuous if the surface current density  $\vec{K}$  is not zero.

#### In-class Activity

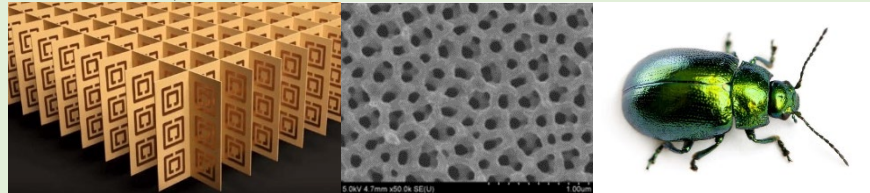
- 2-1. Discuss how an electric or magnetic field interacts with an atom, a molecule, an array of atom/molecule, and a crystalline solid?
- 2-2. Can the superposition principle be valid when you have  $N$ -charged conducting spheres with the same charge  $q$  and diameter placed close together? How about  $N$ -magnetic bars?



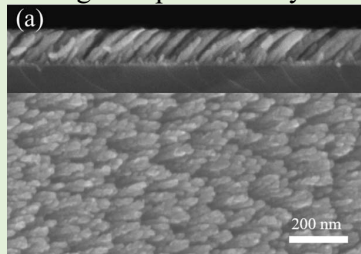
2-3. Derive the displacement current when charging a parallel plate capacitor in an RC circuit.



2-4. How to describe the dielectric and magnetic properties of a metamaterial, a porous material, or the color of a beetle?



2-5. In my lab we are using glancing angle deposition to make nanorod array thin film, see the SEM image below. If the material is dielectric, how can you describe the dielectric function of the material? If the material is magnetic, how can you describe the magnetic permeability?



2-6. How to describe the constitutive relationship for Pockel effect, Kerr effect, Faraday effect, acoustic-optical effect, electro-absorption effect?