Chapter 9
Magnetostatics

9.1 Basic Concepts

According to the Maxwell's equations,

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J}_{total} \end{cases}, \tag{9.1}$$

where $\vec{J}_{total}$ represents the total current density of the system, including the free current density $\vec{J}$ and the bounded current density $\vec{J}_M$, which is the source of the magnetic field. Here

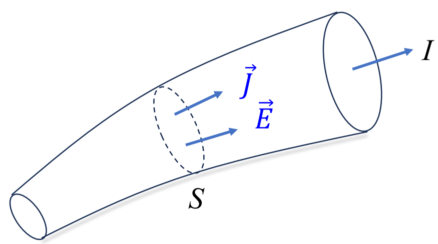

Fig. 9.1 Current and current density.

$$\vec{J} = \frac{I}{A}\hat{n}. \tag{9.2}$$

or,

$$I = \iint_S \vec{J} \cdot d\vec{S}' \tag{9.3}$$

where I is the current flow through a cross section area A and $\hat{n}$ is the current density direction in the cross-section S. For a conductor, $\vec{J}$ follows the Ohms law, $\vec{J} = \sigma\vec{E}$, with σ being the conductivity of the conductor and $\vec{E}$ the electric field inside the conductor. In any isolated material or enclosed object, the total charge of the system shall be conserved, i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{9.4}$$

In magnetostatics, $\frac{\partial \rho}{\partial t} = 0$, thus,

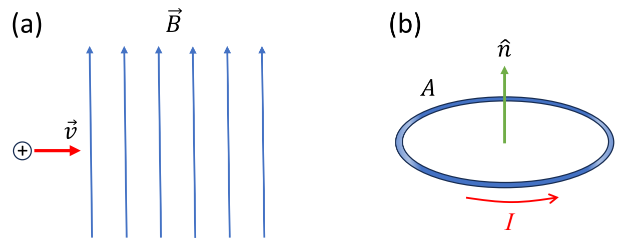

Fig. 9.2 (a) A moving charge goes through an uniform magnetic field. (b) A magnetic dipole.

$$\nabla \cdot \vec{J} = 0, \tag{9.5}$$

which also implies that $\nabla \cdot \vec{E} = 0$, i.e., inside the conductor, there is no accumulated electric charges.

Definition of the magnetic field: a moving charge in a uniform magnetic field $\vec{B}$ as shown in **Figure 9.2a** will experience a magnetic force $\vec{F}_B$,

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{9.6}$$

By measuring both the magnetic force and velocity of the charged particle, in principle one can obtain the magnetic field $|\vec{B}|$, $|\vec{B}| = \frac{|\vec{F}_B|}{q|\vec{v}|\sin\theta}$, where θ is the angle between $\vec{v}$ and $\vec{B}$.

Definition of a magnetic dipole: A current loop shown in **Figure 9.2b** is defined as a magnetic dipole and is the basic unit for the source of magnetic field. It has a magnetic dipole moment $\vec{m}$,

$$\vec{m} = IA\hat{n}, \tag{9.7}$$

Here $\hat{n}$ is the surface normal of the current loop and follows the right-hand rule with respect to the current loop direction. When a magnetic dipole is placed in a magnetic field, it will experience a magnetic force $\vec{F}_B$ which will be discussed in **Lecture 10** and a magnetic torque $\vec{N}_B$,

$$\vec{N}_B = \vec{m} \times \vec{B}. \tag{9.8}$$

9.1.1 Biot-Savart law

The magnetic field is in fact generated by a current carrying wire, and the magnitude is given by the Biot-Savart law. As shown in **Figure 9.3a**, taking a small section $d\vec{l}$ in the current carrying wire, the magnetic field $d\vec{B}$ it generates can be expressed as,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{|\vec{r}|^3}. \tag{9.9}$$

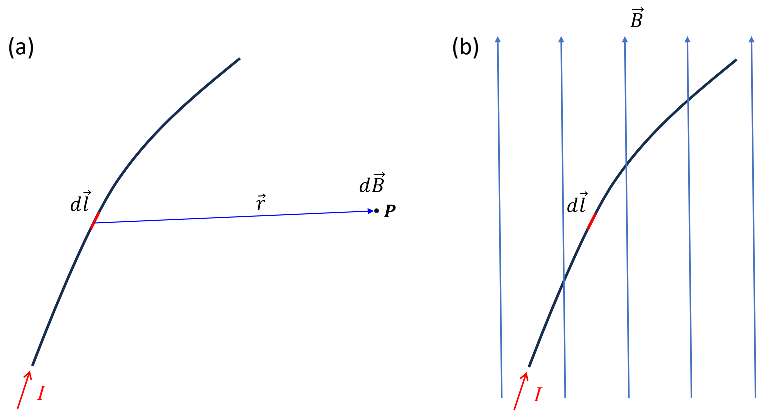

Fig. 9.3 (a) The configuration for Biot-Savart law. (b) The magnetic force on a current-carrying wire in a magnetic field.

For a moving charge with velocity $\vec{v}$, it generates a magnetic field of

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{|\vec{r}|^3}. \tag{9.10}$$

For the entire section of the current carrying wire, the total magnetic field $\vec{B}$ generated at location $\boldsymbol{P}$ can be written as,

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_L \frac{d\vec{l}\times(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}. \tag{9.11}$$

Note that $I = \iint_S \vec{J}\cdot\hat{n}dS'$, where the surface integration is conducted in the cross-section of the wire, and notice that $dS'dl = dV'$, therefore, **Equation 9.11** can be changed to,

$$\vec{B} = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}(\vec{r'})\times(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}dV'. \tag{9.12}$$

For a surface current density $\vec{K}(\vec{r}_s)$ where $\vec{r}_s$ is on the surface of concern, one has,

$$\vec{B} = \frac{\mu_0}{4\pi}\iint_S \frac{\vec{J}(\vec{r}_s)\times(\vec{r}-\vec{r}_s)}{|\vec{r}-\vec{r}_s|^3}dS'. \tag{9.13}$$

Equation 9.12 can be simplified using the following two identities, $\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} = -\nabla\frac{1}{|\vec{r}-\vec{r'}|}$ and $\nabla\times\left(C\vec{F}\right) = C\nabla\times\vec{F} - \vec{F}\times\nabla C$, thus

$$\nabla\times\left[\frac{1}{|\vec{r}-\vec{r'}|}\vec{J}(\vec{r}')\right] = \frac{1}{|\vec{r}-\vec{r'}|}\nabla\times\vec{J}(\vec{r}') - \vec{J}(\vec{r}')\times\nabla\frac{1}{|\vec{r}-\vec{r'}|}. \tag{9.14}$$

The first term in the right-hand side of **Equation 9.14** is zero since $\nabla\times\vec{J}(\vec{r}') = 0$. Therefore,

$$\vec{B} = \frac{\mu_0}{4\pi}\iiint_V \nabla\times\frac{\vec{J}(\vec{r'})}{|\vec{r}-\vec{r'}|}dV' = \frac{\mu_0}{4\pi}\nabla\times\iiint_V \frac{\vec{J}(\vec{r'})}{|\vec{r}-\vec{r'}|}dV'. \tag{9.15}$$

Comparing this expression to the expression for electric field, $\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla \iiint_V \frac{\rho(\vec{r}\prime)}{|\vec{r}-\vec{r}\prime|} dV'$, there are two differences: the source in Equation 9.15 is a vector, the current density $\vec{J}(\vec{r}')$ as compared to scalar charge density $\rho(\vec{r}')$; the operation for **Equation 9.15** is the curl, $\nabla\times$; while for $\vec{E}$, it is a gradient, ∇.

9.1.2 Magnetic force and torque

The force of a current-carrying wire in a magnetic field $\vec{B}$ as shown in in **Figure 9.3b** can be expressed as,

$$d\vec{F}_B = Id\vec{l} \times \vec{B}. \tag{9.16}$$

For a finite section of current-carrying wire, the total magnetic force $\vec{F}_B$ can be written as,

$$\vec{F}_B = \int_L Id\vec{l} \times \vec{B}. \tag{9.17}$$

According to the relationship between the current and current density (**Equation 9.3**), Equation 9.17 can be rewritten as,

$$\vec{F}_B = \iiint_V \vec{J}(\vec{r}') \times \vec{B}(\vec{r}')dV', \tag{9.18}$$

where the volume integration is going through the entire current-carrying wire. And the torque exerted on the wire can be expressed as,

$$\vec{N}_B = \iiint_V \vec{r}' \times \left[\vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\right] dV'. \tag{9.19}$$

9.1.3 Ampere's law

Applying the identity $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ and a curl operator on **Equation 9.15**,

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi}\nabla \times \left[\nabla \times \iiint_V \frac{\vec{J}(\vec{r}\prime)}{|\vec{r}-\vec{r}\prime|} dV'\right]$$

$$= \frac{\mu_0}{4\pi}\nabla\left(\iiint_V \vec{J}(\vec{r}') \cdot \nabla\frac{1}{|\vec{r}-\vec{r}\prime|} dV'\right) - \frac{\mu_0}{4\pi}\iiint_V \vec{J}(\vec{r}')\nabla^2 \frac{1}{|\vec{r}-\vec{r}\prime|} dV'. \tag{9.20}$$

Since $\nabla\frac{1}{|\vec{r}-\vec{r}\prime|} = -\nabla'\frac{1}{|\vec{r}-\vec{r}\prime|}$ and $\nabla^2\frac{1}{|\vec{r}-\vec{r}\prime|} = -4\pi\delta(\vec{r}-\vec{r}')$, above expression can be rewritten as,

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi}\nabla\left(\iiint_V \vec{J}(\vec{r}') \cdot \frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3} dV'\right) + \mu_0\vec{J}(\vec{r}). \tag{9.21}$$

Since $\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + \vec{F}\times(\nabla\times\vec{G}) + (\vec{G}\cdot\nabla)\vec{F} + \vec{G}\times(\nabla\times\vec{F})$, then

$$\nabla\left[\vec{J}(\vec{r}') \cdot \frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}\right] = \left[\vec{J}(\vec{r}')\cdot\nabla\right]\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3} + \vec{J}(\vec{r}') \times \left(\nabla \times \frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}\right)$$

$$+\left(\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}\cdot\nabla\right)\vec{J}(\vec{r}\,') + \frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}\times\left[\nabla\times\vec{J}(\vec{r}\,')\right]$$

The 2nd, 3rd, and 4th terms on the right-hand side of above equation equal to zero, thus,

$$\nabla\left(\iiint_V \vec{J}(\vec{r}\,')\cdot\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}dV'\right) = \iiint_V \left[\vec{J}(\vec{r}\,')\cdot\nabla\right]\frac{\vec{r}-\vec{r}\prime}{|\vec{r}-\vec{r}\prime|^3}dV' = 0.$$

Therefore,

$$\nabla\times\vec{B} = \mu_0\vec{J}(\vec{r}), \tag{9.22}$$

or

$$\oint_L \vec{B}\cdot d\vec{l} = \mu_0\iint_S \vec{J}\cdot\hat{n}dS' = \mu_0\sum I. \tag{9.23}$$

Thus, the Biot-Savart law can derive into Ampere's law.

9.2 Magnetic Potential

9.2.1 Magnetic scalar potential

If $\vec{J}(\vec{r}) = 0$ everwhere, $\nabla\times\vec{B} = 0$. Since $\nabla\cdot\vec{B} = 0$ always holds, then we can write

$$\vec{B}(\vec{r}) = -\nabla\varphi_B(\vec{r}), \tag{9.24}$$

where $\varphi_B(\vec{r})$ is called the magnetic scalar potential and shall satisfy the Laplace equation,

$$\nabla^2\varphi_B(\vec{r}) = 0. \tag{9.25}$$

Therefore, we can solve similar boundary value problems as we did for electrostatics, with different boundary conditions. More details to use $\varphi_B(\vec{r})$ can be found in **Section 11.3**.

9.2.1 Magnetic vector potential

When $\vec{J}(\vec{r}) \neq 0$, $\vec{B}$ cannot be described by the scalar potential. However, by looking back into **Equation 9.15**, we can rewrite the expression for the magnetic field $\vec{B}$ as,

$$\vec{B}(\vec{r}) = \nabla\times\vec{A}(\vec{r}), \tag{9.26}$$

with

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{J}(\vec{r}\prime)}{|\vec{r}-\vec{r}\prime|}dV', \tag{9.27}$$

so that

$$\nabla\times\vec{B} = \nabla\times\left[\nabla\times\vec{A}(\vec{r})\right] = \mu_0\vec{J}. \tag{9.28}$$

Here $\vec{A}(\vec{r})$ is termed as the magnetic vector potential. If $\vec{A}(\vec{r})$ is obtained, then according to **Equation 9.26**, one can obtain $\vec{B}$,

$$\begin{cases} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{cases}. \tag{9.29}$$

Since $\vec{B}$ is the curl of $\vec{A}(\vec{r})$, mathematically, there could be multiple $\vec{A}(\vec{r})$ that can give the same $\vec{B}$, because for any arbitrary function $\chi(\vec{r})$, $\nabla \times [\nabla\chi(\vec{r})] = 0$, therefore,

$$\nabla \times \left[\vec{A}(\vec{r}) + \nabla\chi(\vec{r})\right] = \nabla \times \vec{A}(\vec{r}). \tag{9.30}$$

i.e., theoretically, any vector $\vec{A}'^{(\vec{r})} = \vec{A}(\vec{r}) + \nabla\chi(\vec{r})$ can be treated as possible magnetic vector potentials. However, physically $\vec{A}(\vec{r})$ cannot be arbitrarily chosen, it shall satisfy certain physics principle. Since $\vec{A}(\vec{r})$ should satisfy **Equation 9.28**, applying the identity $\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2\vec{F}$, we have

$$\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A} = \mu_0\vec{J}. \tag{9.31}$$

Therefore, the simplest constraint to $\vec{A}$ is to make $\nabla \cdot \vec{A} = 0$, which means $\nabla^2\chi(\vec{r}) = 0$, so that

$$\nabla^2\vec{A} = -\mu_0\vec{J}. \tag{9.32}$$

This is the Poisson's equation for the vector potential, i.e.,

$$\begin{cases} \nabla \times \vec{A} = \vec{B} \\ \nabla \cdot \vec{A} = 0 \end{cases}. \tag{9.33}$$

Example 9.1 *Find the vector potential for a uniform magnetic field in z-direction.*

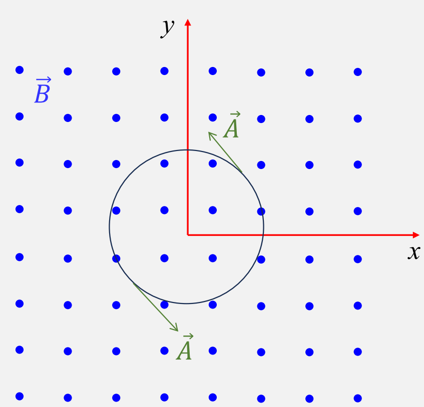

Fig. 9.4 A uniform magnetic field and vector potential.

Discussion: Based on **Equation 9.29**, we shall have

$$\begin{cases} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \\ B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0 \end{cases}$$

There are three possible solutions for above equations,

(1) $A_y = xB_0, A_x = A_z = 0$
(2) $A_x = -yB_0, A_y = A_z = 0$
(3) $A_x = -\frac{1}{2}yB_0, A_y = \frac{1}{2}xB_0, A_z = 0$

All 3 solutions satisfy $\nabla \cdot \vec{A} = 0$.

The last solution can be written as,

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}.$$

Taking a loop integration of $\vec{A}$,

$$\oint_L \vec{A} \cdot d\vec{l} = \iint_S \nabla \times \vec{A} \cdot \hat{n} dS' = \iint_S \vec{B}_0 \cdot \hat{n} dS' = \Phi_B,$$

i.e., it is the magnetic flux through the loop area. Since B_0 is a constant, for a circular loop,

$$\Phi_B = \pi r^2 B_0,$$

while for the loop integration of $\vec{A}$, one has,

$$\oint_L \vec{A} \cdot d\vec{l} = 2\pi r A.$$

Therefore,

$$A = \frac{1}{2}Br.$$

Therefore, the solution $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ is more physically sound, see **Figure 9.4**.

Some expressions for $\vec{A}(\vec{r})$:

For a current-carrying wire,

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_L \frac{d\vec{l}}{|\vec{r}-\vec{l}|}. \quad (9.34)$$

For a surface current density $\vec{K}(\vec{r}_s)$,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iint_S \frac{\vec{K}(\vec{r}_s)}{|\vec{r}-\vec{r}_s|} dS'. \quad (9.25)$$

According to the Posson's equation for $\vec{A}$, each component of $\vec{A}$ shall satisfy the Poisson's equation,

$$\nabla^2 A_i = -\mu_0 J_i. \tag{9.26}$$

Example 9.2 *Find the $\vec{A}(\vec{r})$ of a straight, infinitely long current-carrying wire with a current I.*

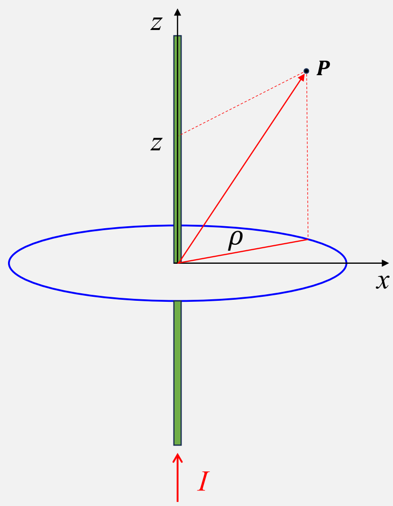

Fig. 9.5 An straight and infinitely long current-carrying wire.

Discussion: For this problem, $J_z = \frac{I}{\pi a^2}$, $J_x = J_y = 0$, therefore $A_x = A_y = 0$. According to **Equation 9.34**, according to **Figure 9.5**, A_z can be expressed as

$$A_z(\rho) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{\rho^2 + z^2}} = \lim_{l \to \infty} \frac{\mu_0 I}{2\pi} ln \frac{\sqrt{\rho^2 + l^2}}{\rho}$$

Neglecting the term of $l \to \infty$, we obtain,

$$A_z(\rho) = -\frac{\mu_0 I}{2\pi} ln\rho$$

The corresponding magnetic field is

$$\begin{cases} B_x = \frac{\partial A_z}{\partial y} = -\frac{\mu_0 I}{2\pi} \frac{y}{\rho^2} \\ B_y = -\frac{\partial A_z}{\partial x} = \frac{\mu_0 I}{2\pi} \frac{x}{\rho^2} \\ B_z = 0 \end{cases}$$

Example 9.3 *Find the magnetic field produced by a circular ring of current.*

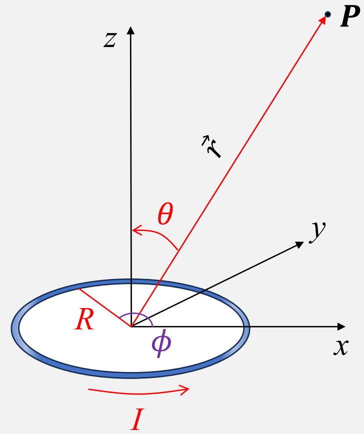

Fig. 9.6 A circular current ring.

Discussion: We can use two different methods to solve the problem.

- **Use the scalar potential to solve the problem**

Look at any location outside the ring, there is no current source in the space, so that we can use **Equation 9.24** to construct a scalar potential $\varphi_B(\vec{r})$, which satisfies Laplace equation. Take the spherical coordinates and due to the geometric azimuthal symmetry of the entire system, the solution for $\varphi_B(\vec{r})$ can be written as,

$$\varphi_B(r,\theta) = \begin{cases} \sum_{l=1}^{\infty} A_l \left(\frac{r}{R}\right)^l P_l(\cos\theta), & r \le R \\ \sum_{l=1}^{\infty} B_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta), & r > R \end{cases}.$$

When $r = R$, the magnetic field shall continue, thus

$$\left.\frac{\partial \varphi_B}{\partial r}\right|_{r^- \to R} = \left.\frac{\partial \varphi_B}{\partial r}\right|_{r^+ \to R}$$

Thus,

$$\left.\sum_{l=1}^{\infty} A_l \frac{l}{R}\left(\frac{r}{R}\right)^{l-1} P_l(\cos\theta)\right|_{r^- \to R} = -\left.\sum_{l=1}^{\infty} B_l \frac{l+1}{r}\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\right|_{r^+ \to R}$$

Therefore,

$$A_l \frac{l}{R} = -B_l \frac{l+1}{r}$$

$$\therefore B_l = -\frac{l}{l+1} A_l$$

In addition, the magnetic field along z-axis can be calculated *via* the Biot-Savart law,

$$B_z(z) = \frac{\mu_0 I}{2} \frac{R^2}{\left(R^2+z^2\right)^{3/2}}$$

Thus,

$$\varphi_z(z) = -\int_0^z B_z(z) dz = -\frac{\mu_0 I}{2} \frac{z}{\left(R^2+z^2\right)^{1/2}}$$

For $z < R$, according to the generating function of Legendre polynomial,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} t^l P_l(x).$$

When $x = 0$, one has

$$\frac{1}{\sqrt{1+t^2}} = \sum_{l=0}^{\infty} t^l P_l(0).$$

$$\therefore \quad \varphi_z(z) = -\frac{\mu_0 I}{2}\sum_{l=1}^{\infty}\left(\frac{z}{R}\right)^l P_l(0)$$

Compare to the general solution for $z < R$,

$$\varphi_B(z, \theta = 0) = \sum_{l=1}^{\infty} A_l \left(\frac{z}{R}\right)^l P_l(1).$$

Since $P_l(1) = 1$, thus

$$A_l = -\frac{\mu_0 I}{2} P_{l-1}(0).$$

Therefore,

$$\varphi_B(r,\theta) = \begin{cases} -\frac{\mu_0 I}{2}\sum_{l=1,3,5,\cdots}^{\infty}\left(\frac{r}{R}\right)^l P_{l-1}(0)P_l(\cos\theta), & r \le R \\ \frac{\mu_0 I}{2}\sum_{l=1,3,5,\cdots}^{\infty}\frac{l}{l+1}\left(\frac{R}{r}\right)^{l+1} P_{l-1}(0)P_l(\cos\theta), & r > R \end{cases}.$$

The reason for l to take the odd integer is because $P_{l-1}(0) = 0$ when l is an even integer. The magnetic field can be found by,

$$\vec{B}(r,\theta) = -\nabla\varphi_B(r,\theta) = -\frac{\partial\varphi_B}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\varphi_B}{\partial\theta}\hat{\theta}.$$

$$\therefore \vec{B}_B(r,\theta) = \begin{cases} -\frac{\mu_0 I}{2}\sum_{l=1,3,5,\cdots}^{\infty}\left(\frac{r}{R}\right)^l P_{l-1}(0)P_l(\cos\theta), & r \le R \\ \frac{\mu_0 I}{2}\sum_{l=1,3,5,\cdots}^{\infty}\frac{l}{l+1}\left(\frac{R}{r}\right)^{l+1} P_{l-1}(0)P_l(\cos\theta), & r > R \end{cases}$$

For $r \gg R$, only take the $l = 1$ term,

$$\varphi_B(r,\theta) \approx \frac{\mu_0}{2\pi}\frac{\pi R^2 I}{r^2}\cos\theta = \frac{\mu_0}{2\pi}\frac{m}{r^2}\cos\theta.$$

Here $m = \pi R^2 I$ is the magnetic dipole moment of the ring current. The corresponding magnetic field,

$$\vec{B}_B(r,\theta) = \frac{\mu_0}{2\pi}\frac{m}{r^3}\cos\theta\,\hat{r} + \frac{\mu_0}{2\pi}\frac{m}{r^3}\sin\theta\,\hat{\theta}.$$

- **Use the vector potential to solve the problem**

The current density for the current ring can be written as,

$\vec{J}(\vec{r}') = \frac{I}{R}\delta(r'-R)\delta(\cos\theta')\hat{\phi} = \frac{I}{R}\delta(r'-R)\delta(\cos\theta')[-\sin\phi'\,\hat{x} + \cos\phi'\hat{y}]$.

We can find the vector potential based on **Equation 9.27**. Since the current source is only confined in the *x*-*y* plane, $A_z = 0$, thus

$$\begin{cases} A_x = -\frac{\mu_0 I}{4\pi R}\iiint_V \frac{\delta(r'-R)\delta(\cos\theta')\sin\phi'}{|\vec{r}-\vec{r}'|_{\phi=0}} r'^2 dr' \, d\Omega' \\ A_y = \frac{\mu_0 I}{4\pi R}\iiint_V \frac{\delta(r'-R)\delta(\cos\theta')\cos\phi'}{|\vec{r}-\vec{r}'|_{\phi=0}} r'^2 dr' \, d\Omega' \end{cases}.$$

Here $d\Omega' = \sin\theta' \, d\theta' d\phi'$ and

$$|\vec{r}-\vec{r}'|_{\phi=0} = [r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi')]^2.$$

The reason we only chose $\phi = 0$ situation is due to the azimuthal symmetry of the system and under $\phi = 0$ condition the math will be simpler.

Since the integrant in the expression for A_x is an odd function with respect to ϕ', the result of the integration shall be zero. Therefore, only A_y is not vanished. Thus,

$$A_y(r,\theta) = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi} \frac{\cos\phi'}{[R^2+r^2-2Rr\sin\theta\cos\phi']^2} d\phi'.$$

Above integration cannot be further solved, but the denominator can be expanded by spherical harmonics or Legendre polynomials.

The components of magnetic field can be written as

$$\begin{cases} B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \, A_y) \\ B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_y) \\ B_\phi = 0 \end{cases}.$$

For $R \gg r$, or $R \ll r$, or $\theta \ll 1$, $A_y(r,\theta)$ can be expanded as

$$A_y(r,\theta) = \frac{\mu_0 IR^2 r\sin\theta}{4(R^2+r^2)^{3/2}}\left[1 + \frac{15R^2r^2\sin^2\theta}{8(R^2+r^2)^2} + \cdots\right].$$

The magnetic field components can be written as,

$$B_r = \frac{\mu_0 IR^2\cos\theta}{2(R^2+r^2)^{3/2}}\left[1 + \frac{15R^2r^2\sin^2\theta}{4(R^2+r^2)^2} + \cdots\right],$$

$$B_\theta = -\frac{\mu_0 IR^2\sin\theta}{4(R^2+r^2)^{\frac{5}{2}}}\left[2R^2 - r^2 + \frac{15R^2r^2\sin^2\theta(4R^2-3r^2)}{8(R^2+r^2)^2} + \cdots\right].$$

Particularly, for $r \gg R$, we have

$$B_r = \frac{\mu_0}{2\pi}\frac{m}{r^3}\cos\theta,$$

$$B_\theta = \frac{\mu_0}{4\pi}\frac{m}{r^3}\sin\theta.$$

In-class Activity

9-1. Show that a magnetic vector potential for two long, straight, parallel wires carrying the same current *I*, in opposite directions is given by $\vec{A} = \frac{\mu_0 I}{2\pi} ln\left(\frac{r_2}{r_1}\right)\hat{n}$, where r_1 and r_2 are the distance from the field point to the wires, and $\hat{n}$ is the unit vector parallel to the wires.

9-2. Given the following set of conductors: an infinitely long straight wire surrounded by a thin cylindrical shell of metal (at radius *b*) arranged co-axially with the wire. The two conductors carry equal but opposite currents *I*. Find the magnetic vector potential for the system.

9-3. Show that the B-field outside of a long straight wire carrying a current *I* is derivable from the scalar potential $\varphi(\vec{r}) = -\frac{1}{2\pi}\theta$ in cylindrical coordinates.